

# Exponential Separation of Quantum and Classical One-Way Communication Complexity for a Boolean Function

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## Abstract

We give an exponential separation between one-way quantum and classical communication complexity for a Boolean function. Earlier such a separation was known only for a relation. A very similar result was obtained earlier but independently by Kerenidis and Raz [KR06]. Our version of the result gives an example in the bounded storage model of cryptography, where the key is secure if the adversary has a certain amount of classical storage, but is completely insecure if he has a similar amount of *quantum* storage.

## 1 Introduction

From a computer science perspective, the main theoretical goal of the field of quantum computing is to exhibit problems where quantum computers are much faster (or otherwise better) than classical computers. Preferably exponentially better. The most famous example, Shor's efficient quantum factoring algorithm, constitutes a separation only if one is willing to believe that efficient factoring is impossible on a classical computer—proving this would of course imply that  $P \neq NP$ . One of the few areas where we can establish *unconditional* exponential separations is the area of communication complexity. Here two parties, Alice with input  $x$  and Bob with input  $y$ , collaborate to solve some computational problem that depends on both  $x$  and  $y$ . They want to do this with minimal communication.

Examples of communication problems where quantum communication gives exponential savings over classical communication were for instance given by Buhrman, Cleve, and Wigderson [BCW98] for zero-error protocols, Raz [Raz99] for bounded-error protocols, Buhrman, Cleve, Watrous, and de Wolf [BCWW01] for simultaneous message passing protocols, and Bar-Yossef, Jayram, and Kerenidis [BJK04] for one-way protocols. The last result establishes an exponential separation for one-way communication: it describes a problem (the Hidden Matching Problem) which can be solved by a quantum message of  $\log n$  qubits, but which cannot be solved with good success probability with much fewer than  $\sqrt{n}$  classical bits of communication. However, their problem is a *relational problem*, where for each  $x$  and  $y$  many possible outputs are considered correct. Establishing a separation for a Boolean *function* was left as an open problem.

In this paper, we give an exponential  $((\log n)^{3/2}$  vs  $\sqrt{n}(\log n)^{1/4})$  quantum-classical separation for one-way communication for a Boolean function. The problem is a variant of a functional problem that was already conjectured to give such a separation by Bar-Yossef et al. As usual in such results, the efficient quantum protocol is quite easy, while the lower bound for classical one-way protocols is harder to show. A very similar separation was obtained earlier but independently by Kerenidis and Raz [KR06] using different

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techniques based on Fourier analysis. The outline of their proof was obtained in the summer of 2004, but was written up in detail when they learned in June 2006 about our independent work (which was not quite finished at the time). We thank Kerenidis and Raz for generously delaying the publication of their proof while we were finishing ours.

Let us briefly point out some differences between our two proofs. The Kerenidis-Raz result is a slightly stronger separation than ours, since the quantum upper bound for their problem is  $\log n$  while ours is  $(\log n)^{3/2}$ . Their proof is heavily based on the analysis of Fourier coefficients. It is self-contained except for an application of the Bonami-Beckner inequality. Our proof is quite different. Intuitively, it shows that if Alice's message was too short, then Bob has hardly any information about a certain string  $z$  that can be computed from  $x$  given also Bob's input. It is based on a result of Talagrand [Tal96] and a large deviation inequality for martingales due to McDiarmid [McD98]. A small advantage of our result is that our quantum protocol has zero-error, while the Kerenidis-Raz variant of the problem gives a separation between bounded-error quantum and classical, but not between zero-error quantum and bounded-error classical.

Another advantage of our proof is that it shows something interesting about the bounded storage model of cryptography. In this model, introduced by Maurer [Mau92], an adversary has full but temporary access to some string  $x$  but can only store a limited amount of information about  $x$ . A and B have a secret key  $K$  available, which they use to derive a common string  $z$  from  $x$  (they only need to store small portions of  $x$  at the time). This string  $z$  is supposed to be almost completely uniformly distributed from the point of view of the adversary, and hence can be used by A and B as a key in a one-time pad communication scheme. The power of this model is that one can show in many cases that the adversary knows very little about the string  $z$  that Alice and Bob derived from  $x$ , *even if the adversary later learns the shared key that was used to derive  $z$*  [ADR02, DM04]. Viewing our communication result in the bounded-storage context, Alice's message is the storage of the adversary, while Bob's input takes the role of the secret key  $K$ . Our result shows that  $z$  can fairly safely be used as a key if the adversary has less than  $\sqrt{n}$  classical storage, while it is completely insecure if the adversary has  $\sqrt{n}$  (or even only polylogarithmic) *quantum* storage.<sup>1</sup>

Finally, let us point out that both results can be modified to give a separation in the simultaneous message passing model between the models of classical communication with shared entanglement and classical communication with shared randomness. Earlier, such a separation was known only for a relational problem [GKRW06], not for a Boolean function.

## 2 The problem and its quantum and classical upper bounds

We assume basic knowledge of quantum computation [NC00] and (quantum) communication complexity [KN97, Wol02]. The partial Boolean function that will give the separation is the following problem, parametrized by a value  $\alpha \leq 1/4$ . It's a modification of the Boolean Hidden Matching Problem from [BJK04].

**Alice:**  $x \in \{0, 1\}^n$

**Bob:**  $\alpha n$  disjoint edges  $e_1 = (i_1, j_1), \dots, e_{\alpha n} = (i_{\alpha n}, j_{\alpha n})$  from  $\binom{[n]}{2}$  and a string  $w \in \{0, 1\}^{\alpha n}$

Define  $z_\ell = x_{i_\ell} \oplus x_{j_\ell}$  and  $z = z_1 \dots z_{\alpha n}$

**Promise:**  $w = z \oplus b^{\alpha n}$  for a bit  $b$

**Function value:**  $b$

There is an easy  $O(\log(n)/\alpha)$ -qubit protocol for this problem that gives the correct output with probability  $1/2$  and claims ignorance otherwise, as follows. Given a uniform superposition over all bits of  $x$  (which takes  $\log n$  qubits), Bob can complete his  $\alpha n$  edges to a perfect matching and measure with the corresponding set of  $n/2$  2-dimensional projectors. With probability  $2\alpha$  he will get one of the edges  $e_\ell = (i_\ell, j_\ell)$  of his input. The state will then collapse to  $(-1)^{x_{i_\ell}}|i_\ell\rangle + (-1)^{x_{j_\ell}}|j_\ell\rangle$ , from which Bob can obtain the bit  $z_\ell = x_{i_\ell} \oplus x_{j_\ell}$  with certainty. XORing this bit with the corresponding bit  $w_\ell$  in his string  $w$  gives the function value  $b$ . The protocol gives Bob  $O(1/\alpha)$  copies of the  $\log n$ -qubit state, so he learns  $b$  with good probability (and knows when he doesn't).

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<sup>1</sup>In a way our result can be viewed as a strong extractor, albeit a rather bad one, where the random seed (the sequence of edges of Bob's input, described below) takes about  $n$  bits.

The following is an easy *classical* upper bound. Suppose Alice uniformly picks a subset of  $d \approx \sqrt{n/\alpha}$  of the bits of  $x$  and sends those to Bob. It is easy to see by the birthday paradox that now with high probability Bob will have both endpoints of at least one of his  $\alpha n$  edges. This enables him to compute the function value  $b$ . To send a uniform subset of  $d$  bits from  $x$ , Alice would need to send about  $d \log n$  bits to Bob, since she needs to describe the  $d$  indices as well as their bitvalues. However, by Newman's theorem [New91], Alice can actually restrict her random choice to picking one out of  $O(n)$  possible  $d$ -bit subsets, instead of one out of all  $\binom{n}{d}$  possible subsets. Hence  $d + O(\log n)$  bits of communication suffice.

In the sections below we show that this classical upper bound is essentially optimal for  $\alpha \approx 1/\sqrt{\log n}$ , which gives the exponential quantum-classical separation.

### 3 Strategy of the proof

We prove a lower bound on classical communication with shared randomness for the problem of the previous section. By the Yao principle, it suffices to prove a lower bound for *deterministic* protocols under the uniform input distribution on the  $x$ 's, the edges, and  $b$  (note that this fixes Bob's second input  $w$ ). Suppose we have a classical deterministic one-way protocol with  $c$  bits of communication and error probability at most  $1/10$  under this distribution. This protocol partitions the set of  $2^n$   $x$ 's into  $2^c$  sets  $A_1, \dots, A_{2^c}$ , one for each possible message. At least half of the  $x$ 's must occur in sets of size at least  $2^{n-c-1}$ , since the smaller sets together contain fewer than  $2^c \cdot 2^{n-c-1} = 2^{n-1}$   $x$ 's. Hence there must be at least one set  $A$  that contains at least  $2^{n-c-1}$   $x$ 's and has error at most  $1/5$ , otherwise the overall error would be larger than  $1/10$ . Hereafter we will analyze this set  $A$ .

From Bob's point of view the following happens when he receives the message corresponding to  $A$ :  $\alpha n$  disjoint edges  $(i_\ell, j_\ell)$ ,  $\ell \in [\alpha n]$ , uniformly picked from  $\binom{[n]}{2}$  are given, and an unknown  $x$  is picked uniformly from  $A$ . As before, let  $z_\ell = x_{i_\ell} \oplus x_{j_\ell}$  and  $z = z_1 \dots z_{\alpha n}$ . Bob needs to figure out whether his second input equals  $z \oplus 0^{n/4}$  or  $z \oplus 1^{n/4}$ . We will use capital letters to denote the corresponding random variables. Our goal here is to show that  $Z$  is close to uniformly distributed when the edges are known but  $x$  is not. Suppose we can show that if the communication  $c$  is "small", then  $Z$  is more or less uniform: the total variation distance is  $d(Z, U_{\alpha n}) = \frac{1}{2} \sum_{z \in \{0,1\}^{\alpha n}} |\Pr[Z = z] - 2^{-\alpha n}| \leq \delta$  for some small  $\delta$  (this is the bulk of the proof below).<sup>2</sup> Then also  $d(Z \oplus 0^{\alpha n}, U_{\alpha n}) \leq \delta$  and  $d(Z \oplus 1^{\alpha n}, U_{\alpha n}) \leq \delta$ , and hence by the triangle inequality

$$d(Z \oplus 0^{\alpha n}, Z \oplus 1^{\alpha n}) \leq d(Z \oplus 0^{\alpha n}, U_{\alpha n}) + d(Z \oplus 1^{\alpha n}, U_{\alpha n}) \leq 2\delta.$$

But distinguishing between the two distributions  $Z \oplus 0^{\alpha n}$  and  $Z \oplus 1^{\alpha n}$  is exactly what Bob needs to do to determine  $b$ . It is well known that distinguishing between two distributions with variation distance  $2\delta$  can be done with probability at most  $1/2 + \delta$ . Accordingly, if  $c$  is "small" then the success probability will be close to  $1/2$ . Conversely, since Bob's success probability on the set  $A$  is at least  $4/5$ ,  $c$  must have been large.

### 4 How biased are the bits of $Z$ ?

We will analyze the distribution of  $Z$ , which depends on the known edges  $e_1 = (i_1, j_1), \dots, e_{\alpha n} = (i_{\alpha n}, j_{\alpha n})$  as well as the unknown  $x \in A$ . Intuitively, if  $c$  is small (i.e.  $A$  is large), for most strings  $z \in \{0,1\}^n$  we should have  $\Pr[Z = z] \approx 2^{-\alpha n}$  and hence  $d(Z, U_{\alpha n})$  is small. Proving this will be quite technical.

We view the edges as being picked one by one. Since  $A$  is quite large, for most  $(i, j)$ -pairs roughly equally many  $x$ 's should have  $x_i \oplus x_j = 1$  as have  $x_i \oplus x_j = 0$ . Thus we expect the first bit  $Z_1$  to be close to uniformly distributed when  $x$  is picked uniformly from  $A$ . Similarly, we would like the later bits  $Z_\ell$  to be more or less uniform when conditioned on values  $Z_1 = z_1, \dots, Z_{\ell-1} = z_{\ell-1}$  for the earlier edges. More formally, once  $(i_1, j_1), \dots, (i_{\ell-1}, j_{\ell-1})$  and  $z_1, \dots, z_{\ell-1}$  have been fixed, we define the " $\ell$ th bias" by

$$\beta_\ell = \Pr_{x \in A} [Z_\ell = 1 \mid Z_1 = z_1, \dots, Z_{\ell-1} = z_{\ell-1}] - 1/2.$$

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<sup>2</sup>Note that we include a factor of  $1/2$  in our definition of total variation distance. This means that the distance lies in the interval  $[0, 1]$ , and if distributions  $P$  and  $Q$  have distance  $\delta$ , the probability of any event cannot change by more than  $\delta$ , i.e.,  $|\Pr_P[E] - \Pr_Q[E]| \leq \delta$  for any event  $E$ .

This is a random variable, depending on the choice of  $(i_\ell, j_\ell)$ . It is positive if  $Z_\ell$  is biased towards 1, and negative if  $Z_\ell$  is biased towards 0.

Fixing the first  $\ell - 1$  edges and conditioning on their bitvalues  $Z_1 = z_1, \dots, Z_{\ell-1} = z_{\ell-1}$  will shrink the set of possible  $x$ 's. Let  $A_\ell$  be the subset of  $A$  that is still consistent. Initially we have  $|A_1| = |A| \geq 2^{n-c-1}$ . When we pick the next edge  $(i_\ell, j_\ell)$  and its value  $z_\ell$ , the new set  $A_{\ell+1}$  will be smaller by a factor  $1/2 + \beta_\ell$  if  $z_\ell = 1$  and by a factor  $1/2 - \beta_\ell$  if  $z_\ell = 0$ . Hence we expect the set to shrink by a factor of about two for each new edge and bitvalue for that edge, i.e.,  $|A_\ell| \geq 2^{n-c-\ell}$ . We have

$$|A_\ell| = |A| \cdot \Pr[Z_1 = z_1, \dots, Z_{\ell-1} = z_{\ell-1}] = |A| \cdot \prod_{i=1}^{\ell-1} (1/2 - (-1)^{z_i} \beta_i),$$

and in particular

$$\Pr[Z = z] = \prod_{\ell=1}^{\alpha n} \Pr[Z_\ell = z_\ell \mid Z_1 = z_1, \dots, Z_{\ell-1} = z_{\ell-1}] = \prod_{\ell=1}^{\alpha n} (1/2 - (-1)^{z_\ell} \beta_\ell) = \frac{|A_{\alpha n+1}|}{|A|}.$$

Hence showing that  $Z$  is close to uniformly distributed is equivalent to showing that  $|A_{\alpha n+1}|/|A| \approx 2^{-\alpha n}$  with high probability.

We use a result of Talagrand [Tal96] to relate the expected squared bias  $\beta_\ell^2$  to the size of the set  $A_\ell$ .

**Lemma 1** ([Tal96], **Eq. (2.9)**) *There is an absolute constant  $K \geq 1$  such that for all  $A \subseteq \{0, 1\}^n$*

$$\sum_{i,j \in [n], i \neq j} \beta_{ij}^2 \leq K \left( \log \frac{K2^n}{|A|} \right)^2,$$

where  $\beta_{ij} = \mathbb{E}_{x \in A}[x_i \oplus x_j - 1/2]$ .

This will allow us to establish a bound showing that  $\beta_\ell$  is probably quite small if the set  $A_\ell$  hasn't shrunk too fast. We allow some more shrinking than we expect: note the ' $3c$ ' instead of ' $c$ ' in the exponent below.

**Corollary 2** *There is an absolute constant  $\gamma > 0$  such that if  $|A_\ell| \geq 2^{n-3c-\ell}$ , then*

*(1)  $\mathbb{E}[\beta_\ell^2] \leq \gamma(c/n)^2$  and (2)  $\Pr[|\beta_\ell| \geq \varepsilon] \leq \gamma(\frac{c}{n\varepsilon})^2$ .*

**Proof.** Note that fixing a bitvalue for the parity of an edge means that the two bits in that edge behave as one bit. Accordingly, we can view the set  $A_\ell$  as a set of strings of length  $m = n - (\ell - 1)$  bits. We can upper bound the sum of biases over all possible new edges (excluding ones touching earlier edges) by the sum over all possible edges (including ones touching earlier edges):

$$\sum_{i_\ell, j_\ell \notin \{i_1, \dots, i_{\ell-1}, j_1, \dots, j_{\ell-1}\}} \beta_{i_\ell, j_\ell}^2 \leq \sum_{i, j \in [m], i \neq j} \beta_{ij}^2 \leq O(c^2),$$

where the last inequality is by applying Lemma 1 to  $A_\ell$ . Dividing by the number  $\binom{n-2(\ell-1)}{2} = \Theta(n^2)$  of possible new edges proves part (1). Part (2) now follows from Chebyshev's inequality.  $\square$

There is a threat of circularity in our proof. On the one hand we need to assume that the sets  $A_\ell$  are not too small in order to show that the biases  $\beta_\ell$  are not too large (via Corollary 2). But on the other hand we need to show that the biases are not too large in order to be able to conclude that  $A_\ell$  is not too small. To deal with this problem, below we give a proof in two "passes". The first pass is quite coarse-grained and shows that (with high probability) the sets  $A_\ell$  won't shrink by a factor of  $2^{-2c}$  more than what we expect. This allows us to apply Corollary 2 to each of the  $\alpha n$  biases during the second pass. In this second, more fine-grained pass we actually show that  $d(Z, U_{\alpha n})$  is small.

## 5 First pass: The sets $A_\ell$ probably don't shrink too much

We can only use Corollary 2 if the condition  $|A_\ell| \geq 2^{n-3c-\ell}$  is satisfied. We now show that with high probability this is indeed the case for each  $\ell$  simultaneously. The proof uses the following concentration result from [McD98].

**Lemma 3 ([McD98], special case of Thm. 3.7)** *Let  $S_1, \dots, S_k$  be bounded random variables satisfying  $E[S_j | S_1 = s_1, \dots, S_{j-1} = s_{j-1}] = 0$  for all  $1 \leq j \leq k$  and all values  $s_1, \dots, s_k$ . Then for all  $t, v \geq 0$*

$$\Pr \left[ \sum_{j=1}^k S_j \geq t \right] \leq e^{-t^2/2v} + \Pr \left[ \sum_{j=1}^k S_j^2 \geq v \right].$$

**Lemma 4** *There is a constant  $\delta_0 > 0$  such that for all  $0 < \delta \leq \delta_0$ , if  $\alpha = \delta^2/(4\sqrt{\ln n})$  and  $c = \delta\sqrt{n}(\ln n)^{1/4}$  then with probability 99/100 over all choices of  $e_1, \dots, e_{\alpha n}$  and  $z = z_1, \dots, z_{\alpha n}$  the following holds: for each  $\ell \in [\alpha n]$  we have  $|A_\ell| \geq 2^{n-3c-\ell}$  and  $|\beta_\ell| \leq 1/4$ .*

**Proof.** Note that

$$|A_\ell| = |A| \cdot \Pr[Z_1 = z_1, \dots, Z_{\ell-1} = z_{\ell-1}] = |A| \cdot \prod_{i=1}^{\ell-1} (1/2 - (-1)^{z_i} \beta_i) \geq 2^{n-c-\ell} \prod_{i=1}^{\ell-1} (1 - (-1)^{z_i} 2\beta_i).$$

Define  $S_i = -(-1)^{z_i} 2\beta_i$ . For the lower bound on  $A_\ell$  it thus suffices to lower bound  $\prod_{i=1}^{\ell-1} (1 + S_i)$  by  $2^{-2c}$ . Taking logarithms, we need to show for each  $\ell$

$$\sum_{i=1}^{\ell-1} \log(1 + S_i) \geq -2c. \tag{1}$$

Let us divide the  $\alpha n$   $\ell$ s into blocks of size  $c$ , i.e., for  $1 \leq k \leq \alpha n/c$  define the  $k$ th block  $B_k = \{(k-1)c + 1, \dots, kc\}$  (we ignore rounding for simplicity). Let  $E_k$  be the following event:

- (a)  $|\beta_i| \leq 1/4$  for each  $i \in B_k$  and
- (b)  $\sum_{i \in B_k} \log(1 + S_i) \geq -c^2/\alpha n$ .

We will show below in Claim 5 that for all  $k$ ,  $\Pr[\neg E_k \mid E_1, \dots, E_{k-1}] \leq c/100\alpha n$ . This implies

$$\Pr[\neg(E_1, \dots, E_{\alpha n/c})] \leq \sum_{k=1}^{\alpha n/c} \Pr[\neg E_k \mid E_1, \dots, E_{k-1}] \leq \frac{\alpha n}{c} \cdot \frac{c}{100\alpha n} = \frac{1}{100}.$$

If  $E_1, \dots, E_{\alpha n/c}$  all hold, then from (b) for all  $k$  we have  $\sum_{i=1}^{k \cdot c} \log(1 + S_i) \geq -k \cdot c^2/\alpha n \geq -c$  and in particular Eq. (1) holds whenever  $\ell - 1$  is a multiple of  $c$ . For the other  $\ell$  pick  $k$  such that  $\ell - 1 \in B_{k+1}$  and note that thanks to (a) we have  $\log(1 + S_i) \geq \log(1 - 2(1/4)) = -1$  and hence

$$\sum_{i=1}^{\ell-1} \log(1 + S_i) = \sum_{i=1}^{kc} \log(1 + S_i) + \sum_{i=kc+1}^{\ell-1} \log(1 + S_i) \geq -c + \sum_{i=kc+1}^{\ell-1} -1 \geq -2c.$$

**Claim 5** *For all  $1 \leq k \leq \alpha n/c$ , we have  $\Pr[\neg E_k \mid E_1, \dots, E_{k-1}] \leq c/100\alpha n$ .*

**Proof.** Let  $\ell_1 = (k-1)c$  be the last index in  $B_{k-1}$  and condition on  $E_1, \dots, E_{k-1}$ . This means that  $|A_{\ell_1+1}| \geq 2^{n-2c-\ell_1-1}$ . Let  $F_i$  denote the event that (a) holds for  $i \in B_k$ , i.e.,  $|\beta_i| \leq 1/4$ . We want to show  $\Pr[\neg F_i \mid F_1, \dots, F_{i-1}] \leq 1/500\alpha n$  for  $i \in B_k$ . This will imply that (a) fails to hold only with

probability at most  $c/500\alpha n$ . If  $|\beta_{\ell_1+1}|, \dots, |\beta_{\ell_1+i-1}| \leq 1/4$  then as before  $\sum_{j=\ell_1+1}^{i-1} \log(1 + S_j) \geq -c$  and hence  $|A_{\ell_1+i}| \geq |A_{\ell_1+1}| \cdot 2^{-i-c} \geq 2^{n-3c-(\ell_1+i)}$ . We can now apply Corollary 2(part 2) to show

$$\Pr[|\beta_{\ell_1+i}| > 1/4] \leq \gamma \left( \frac{4c}{n} \right)^2 = 16\gamma\delta^2 \frac{\sqrt{\ln n}}{n} \leq \frac{1}{500\alpha n},$$

where we use  $\alpha = \delta^2/(4\sqrt{\ln n})$  and  $c = \delta\sqrt{n}(\ln n)^{1/4}$  and choose  $\delta_0$  small enough.

Now, assuming (a) holds for each  $i \in B_k$  and hence the conditions of Corollary 2 hold for each  $i \in B_k$ , we will show that (b) holds for  $B_k$  with probability at least  $1 - 4c/500\alpha n$ , which will imply the claim.

Note that  $\log(1 + S_i) \geq S_i - 2S_i^2$  if  $|\beta_i| \leq 1/4$  and hence

$$\sum_{i \in B_k} \log(1 + S_i) \geq \sum_{i \in B_k} S_i - 2 \sum_{i \in B_k} S_i^2 = \sum_{i \in B_k} S_i - 8 \sum_{i \in B_k} \beta_i^2. \quad (2)$$

We first bound the second term of the righthand side. Corollary 2 implies

$$\mathbb{E} \left[ \sum_{i \in B_k} \beta_i^2 \right] \leq c \cdot \gamma(c/n)^2 = \gamma c^3/n^2.$$

Let  $v = c^2/2\alpha n$ ; this is half of what (b) allows us to lose. By Markov's inequality

$$\Pr \left[ 8 \sum_{i \in B_k} \beta_i^2 > v \right] \leq 16\alpha\gamma c/n \leq c/500\alpha n$$

for sufficiently large  $n$ .

Now for the first term in the righthand side of Eq. (2). Conditioning on event (a) changes the (a priori uniform) distribution on the  $z_i$  for the  $i \in B_k$  by at most  $c/500\alpha n$  in total variation distance. This means that if we bound  $\Pr[\sum_{i \in B_k} S_i \leq -v]$  under the assumption that the  $z_i$  are uniform, the true probability will change by at most  $c/500\alpha n$  (see footnote 2). If the  $z_i$  are uniform then the condition of Lemma 3 holds for each  $S_i$ : the conditional expectations are all 0, because the sign of  $S_i$  is  $+$  or  $-$  with equal probability. Recall that  $S_i^2 = 4\beta_i^2$  and  $v = c^2/2\alpha n$ . Hence by Lemma 3 (with  $t = v$ ), if the  $z_i$  are uniform

$$\Pr \left[ \sum_{i \in B_k} S_i < -v \right] \leq e^{-v/2} + \Pr \left[ 4 \sum_{i \in B_k} \beta_i^2 \geq v \right] \leq 1/n + c/500\alpha n \leq 2c/500\alpha n.$$

Putting everything together we upper bound the probability that (b) fails for the  $k$ th block:

$$\Pr \left[ \sum_{i \in B_k} \log(1 + S_i) < -c^2/\alpha n = -2v \right] \leq \Pr \left[ \sum_{i \in B_k} S_i < -v \right] + \Pr \left[ 8 \sum_{i \in B_k} \beta_i^2 > v \right] \leq 4c/500\alpha n.$$

This concludes the proof of Claim 5. □

This concludes the proof of Lemma 4. □

## 6 Second pass: $Z$ is close to uniform

We now prove the main result, which implies the  $\tilde{\Omega}(n^{1/2})$  lower bound on classical one-way communication.

**Theorem 6** *There is a constant  $\delta > 0$  such that if  $c = \delta\sqrt{n}(\ln n)^{1/4}$  and  $\alpha = \frac{\delta^2}{4\sqrt{\ln n}}$ , then  $d(Z, U_{\alpha n}) \leq 1/10$ .*

Of course this theorem also holds if the communication of the classical protocol is  $c < \delta n^{1/2}(\ln n)^{1/4}$ , since we can always add dummy bits to a shorter message to make its length equal to exactly that value.

**Proof.** We rewrite the total variation distance:

$$\begin{aligned} d(Z, U_{\alpha n}) &= \frac{1}{2} \sum_{z \in \{0,1\}^{\alpha n}} |\Pr[Z = z] - 2^{-\alpha n}| \\ &= \frac{1}{2} 2^{-\alpha n} \sum_{z \in \{0,1\}^{\alpha n}} \left| \prod_{\ell=1}^{\alpha n} (1 - (-1)^{z_\ell} 2\beta_\ell) - 1 \right| \\ &= \frac{1}{2} \mathbb{E}_z \left[ \left| \prod_{\ell=1}^{\alpha n} (1 - (-1)^{z_\ell} 2\beta_\ell) - 1 \right| \right], \end{aligned}$$

where  $\mathbb{E}_z$  denotes the expectation over uniform  $z$ . By Lemma 4, with probability 99/100, for each  $\ell \in [\alpha n]$  we have  $|A_\ell| \geq 2^{n-3c-\ell}$  and  $|\beta_\ell| \leq 1/4$ . Let us call this event  $E$ . Then

$$d(Z, U_{\alpha n}) \leq \Pr[E] \cdot d(Z|_E, U_{\alpha n}) + \Pr[\neg E] \cdot d(Z|_{\neg E}, U_{\alpha n}) \leq \frac{99}{100} \cdot d(Z|_E, U_{\alpha n}) + \frac{1}{100}.$$

Note that conditioning on  $E$  will change the (a priori uniform) distribution on the  $z$ . However, the total variation distance between the conditioned distribution  $z|_E$  of the  $z$  and the uniform distribution is at most 1/100, since the event  $E$  on which it is conditioned has probability at least 99/100. Hence

$$\begin{aligned} d(Z|_E, U_{\alpha n}) &= \mathbb{E}_{z|_E} \left[ \left| \prod_{\ell=1}^{\alpha n} (1 - (-1)^{z_\ell} 2\beta_\ell) - 1 \right| \right] \\ &\leq \mathbb{E}_z \left[ \left| \prod_{\ell=1}^{\alpha n} (1 - (-1)^{z_\ell} 2\beta_\ell) - 1 \right| \right] + 1/100 \\ &= \mathbb{E}_z \left[ \left| 2^{\sum_{\ell=1}^{\alpha n} \log(1+S_\ell)} - 1 \right| \right] + 1/100, \end{aligned}$$

where  $S_\ell = -(-1)^{z_\ell} 2\beta_\ell$  as in the previous section. We thus need to show that  $\sum_{\ell=1}^{\alpha n} \log(1+S_\ell)$  is usually very close to 0. When conditioned on event  $E$  we have  $|\beta_\ell| \leq 1/4$  and hence

$$\frac{S_\ell}{\ln 2} \geq \log(1+S_\ell) \geq S_\ell - 2S_\ell^2.$$

It thus suffices to show that with high probability, both  $|\sum_{\ell=1}^{\alpha n} S_\ell|$  and  $\sum_{\ell=1}^{\alpha n} S_\ell^2$  are small. This can be done in the same way as in the proof of Claim 5, using this time that  $\mathbb{E}[\sum_{\ell=1}^{\alpha n} \beta_\ell] \leq \gamma \alpha c^2/n = o(1)$ .  $\square$

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